Schrödinger operators with matrix potentials. Transition from the absolutely continuous to the singular spectrum.

S. Molchanov, B. Vainberg
Dept. of Mathematics, University of North Carolina at Charlotte, Charlotte, NC 28223, USA

Abstract

It is proven that the absolutely continuous spectrum of matrix Schrödinger operators coincides (with the multiplicity taken into account) with the spectrum of the unperturbed operator if the (matrix) potential is square integrable. The same result is also proven for some classes of slower decaying potentials if they are smooth.

Keywords Schrödinger operator, absolutely continuous spectrum, L^2 conjecture.

1. Introduction. The goal of this work is to prove the "spectral L^2- conjecture" for Schrödinger operators with matrix potentials. We shall consider the Hamiltonian

$$H\psi = -\frac{d^2\psi}{dx^2} + v(x)\psi, \qquad x \geq 0, \tag{1}$$

acting in the space $L^2(R_+)$ of vector functions $\psi(x) = [\psi_1, ..., \psi_n]^t(x)$ with the boundary condition $\psi'(0) = 0$. The potential v here is a symmetric $n \times n$ matrix. Since the absolutely continuous (a.c.) spectrum of H does not depend on the type of the boundary condition (b.c.), we restrict ourself to the case of the Neumann b.c.. We could consider simultaneously operators on the half axis and on the whole axis. The multiplicity of the spectrum will be doubled in the second case. For the sake of simplicity we decided to focus on the case of the semiaxis. The results below can be also easily extended to the case of general canonical systems and lattice operators.

The spectral L^2- conjecture concerns the minimal decay of the potential at infinity which still guarantees the existence of the rich a.c. spectrum of the operator. S. Kotani, N. Ushiroya [6] and F. Delyon, B. Simon, B. Souillard [5] described the bifurcation from a.c. to pure point (p.p.) spectrum for scalar Schrödinger operators with modulated (decaying) random potentials of the form:

$$v(t, \omega) = \frac{\xi(t, \omega)}{1 + |t|^{\alpha}},$$

were $\xi(t,\omega)$, $t \in R$, is a Markov homogeneous ergodic bounded process on the probability space (Ω, F, P). They proved that the random Schrödinger operator $H(\omega) = -\frac{d^2}{dt^2} + v(t,\omega)$ in L^2 on the whole axis has dense p.p. spectrum on $[0,\infty)$ with probability one ($P-$ a.s.) if $\alpha < 1/2$, and its spectral measure on $[0,\infty)$ is $P-$ a.s. pure a.c. with multiplicity two if $\alpha > 1/2$. It turns out that the same type of bifurcation is valid in the deterministic case and for potentials which are not necessarily decaying as a power. This bifurcation is a consequence of the spectral L^2- conjecture which states that if $v \in L^2(R)$ then $\Sigma_{ac}(H) = [0,\infty)$ and the a.c. component μ_{ac} of the spectral measure of operator H is essentially supported on $[0,\infty)$, i.e. $\mu_{ac}(\Gamma) > 0$ for any Borel set Γ, $|\Gamma| > 0$. Note that $v \in L^2(R)$ if $\alpha < 1/2$ for the random potential above, and $v \notin L^2(R)$ if $\alpha > 1/2$. So, these results on random operators show the exactness of the conjecture.

The spectral L^2- conjecture for the scalar Schrödinger operators was justified in 1999 by Deift and Killip [4]. In 2001, the authors of this paper offered a different approach [9] which allowed us to get a sequence of conditions on the potential v such that under each of them the essential support of the a.c. spectrum of the operator H on the semiaxis R_+ is $[0,\infty)$. These conditions are related to the boundedness of the first KdV integrals, and the first of those conditions is $v \in L^2(R_+)$. Other conditions allow the potential to decay slower if it is smooth.

This paper is devoted to the extension of our results for scalar operators to the matrix Schrödinger operators. We shall show that the spectral measure of the operator (1) has the a.c. component, which has the multiplicity n and is essentially supported on $[0,\infty)$, if for some $p \geq 0$ the following functional is bounded:

$$J_p(v) := \int_{-\infty}^{\infty} \left(||v^{(p-1)}(x)||^2 + ||v(x)||^{p+1} \right) dx, \tag{2}$$

It is assumed here that v is extended by zero for $x < 0$. This implies that $v^{(j)}(0) = 0$ for $j < p-1$. In fact, the latter restriction is not essential, and one can easily show that the main result remains valid if the lower limit in (2) is replaced by zero. Examples of the condition (2) are

$$||v(x)|| \quad \in \quad L^1, \qquad p=0; \qquad ||v(x)|| \in L^2, \qquad p=1;$$

$$\int_{-\infty}^{\infty} \left(||\dot{v}(x)||^2 + ||v(x)||^3 \right) dx \quad < \quad \infty, \qquad p=2, \quad \text{etc}$$

The result above is the simplest form of the generalized L^2-conjecture for matrix operators. The case when $v(x) = L + v_0(x)$, L is a constant matrix, $\|v_0\| \in L^2(R_+)$, will be published elsewhere. The main feature of the latter model is the different multiplicity of the a.c. component on the different intervals of the spectral axis. These intervals are defined by the eigenvalues of the matrix L.

The main scheme of the proof of the matrix L^2-conjecture remains the same as in our work for the scalar case. We approximate the operator H by

operators H_s with matrix potentials $v_s \in C_0^\infty(R_+)$ such that $v_s \to v$ in $L^2(R_+)$ as $s \to \infty$, and we construct the spectral measure $\mu(d\lambda)$ of operator H as a weak limit of spectral measures $\mu_s(d\lambda)$ of operators H_s. Then we express the measures $\mu_s(d\lambda)$ through the scattering data and use trace identities to prove that the limiting measure $\mu(d\lambda)$ has the a.c. component of the same type as measures $\mu_s(d\lambda)$. However, the implementation of this scheme in the matrix case requires to overcome some difficulties related to non-commutativity of matrix multiplication. We shall also simplify some arguments used in ([9]). So, we hope that this publication will make the proof much more transparent even in the scalar case.

In the next two sections of this paper we shall give a review of some preliminary facts which we need for the proof, and we shall construct the spectral measure for the operator (1) as a limit of spectral measures of operators with compactly supported potentials. Until a specific condition on the potential is imposed, we assume that the potential satisfies the Birman condition:

$$\int_x^{x+1} \|v(z)\|\, dz \le c_0 < \infty \quad \text{for all } x \ge 0, \tag{3}$$

which implies that the spectrum of H is bounded from below, and therefore, the operator H is essentially self adjoint. Most of the results included in the sections 2 and 3 can be found in [1], [2], [3], [7], [8].

The main results will be proved in the sections 4 and 5 of the paper.

2. Symplectic structure. Green's matrix. We assume here that (3) holds. The equation

$$H\psi = -\ddot{\psi} + v\psi = \lambda\psi, \qquad x \in R_+, \tag{4}$$

can be written in the canonical form. Put $\psi' = p$ and $Y = (\psi, p)^t$, then

$$-J\dot{Y} = VY + \lambda QY, \tag{5}$$

where

$$J = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}, \qquad V = \begin{bmatrix} v & 0 \\ 0 & -I \end{bmatrix}, \qquad Q = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}. \tag{6}$$

Here I is the identity $n \times n$ matrix, 0 is $n \times n$ matrix of zeroes and $-J^2$ is the identity $2n \times 2n$ matrix.

We denote by $M_\lambda(0,x)$ the transfer matrix for the system (5) (or (4)) which is given by matrix equation

$$-J\dot{M}_\lambda = (V + \lambda Q)M_\lambda, \quad M_\lambda(0,0) = I. \tag{7}$$

It is well known that $M_\lambda(0,x) \in Sp(2n) \subset SL(2n,R)$. The symplectic group $Sp(2n)$ consists of matrices M which preserve the skew-symmetric product $\langle x, y\rangle = (Jx, y)$, i.e. $M \in Sp(2n)$ if $\langle Mx, My\rangle = \langle x, y\rangle$. An equivalent characteristics of symplectic matrices is the identity

$$M^t J M = J. \tag{8}$$

One can easily check that $M_\lambda(0,x)$ satisfies (8) by differentiating both sides of (8) with $M = M_\lambda$ and using (7) and (6). Thus, for any x,

$$M_\lambda(0,x) \in Sp(2n). \tag{9}$$

If $M \in Sp(2n)$ is presented in the block form

$$M = \begin{bmatrix} P & Q \\ R & T \end{bmatrix}$$

with $n \times n$ matrices P, Q, R, T, then the identity (8) implies "pseudo commutativity":

$$P^t R = R^t P, \quad T^t Q = Q^t T$$

and the "matrix unimodularity":

$$T^t P - Q^t R = I.$$

Since $M^t \in Sp(2n)$ then also

$$PQ^t = QP^t, \quad RT^t = TR^t.$$

We shall need the concept of the Lagrangian plane. Linear n-dimensional subspace $\pi \in R^{2n}$ is called the Lagrangian plane if $\langle x, y\rangle = (Jx, y) = 0$ for any $x, y \in \pi$. If π is a Lagrangian plane and $M \in Sp(2n)$ then $M\pi$ is also a Lagrangian plane. In particular, the transfer matrix maps any Lagrangian plane into a Lagrangian plane. It will be used in the following context. Let $u_\lambda(x)$ be a $n \times n$ matrix solution of the system (4). Consider its Cauchy data $[u_\lambda, \dot{u}_\lambda]^t$. Let $\pi(x)$ be the span of the columns of the latter matrix. If $\pi(x)$ is a Lagrangian plane for one value of x then $\pi(x)$ is a Lagrangian plane for any x.

The Wronskian of two $n \times n$ matrices is a $n \times n$ matrix defined by the formula

$$W(u,v) = \begin{vmatrix} u(x) & v(x) \\ \dot{u}(x) & \dot{v}(x) \end{vmatrix} = \dot{v}^t(x)u(x) - v^t(x)\dot{u}(x). \tag{10}$$

The Wronskian can also be written in the form

$$W(u,v) = [v^t, \dot{v}^t]J\begin{bmatrix} u \\ \dot{u} \end{bmatrix}. \tag{11}$$

The same arguments as in the scalar case imply that the Wronskian $W(u,v)$ does not depend on x if the matrices u, v satisfy the equation (4).

Let us consider the following Sturm-Liouville problem for the equation (4)

$$(H-\lambda)\psi = 0, \quad x_1 < x < x_2; \quad [\psi, \dot{\psi}]^t_{x=x_1} \in \pi_1, \quad [\psi, \dot{\psi}]^t_{x=x_2} \in \pi_2, \tag{12}$$

where π_1, π_2 are two Lagrangian planes. Conservation of the Wronskian allows us to construct the Green function for that problem. Let u_λ (v_λ) be a matrix whose columns are solutions of the Cauchy problem for the equation (4) with the Cauchy data at $x = x_1$ ($x = x_2$) which form a basis in π_1 (π_2, respectively).

Lemma 1 *The determinant* $\det W(u_\lambda, v_\lambda)$ *is equal to zero if and only if* λ *is an eigenvalue of the problem (12). If* $\det W \neq 0$ *then the matrix*

$$G_\lambda(x,\xi) = \left\{ \begin{array}{ll} -u_\lambda(x)[W(u_\lambda, v_\lambda)]^{-1} v_\lambda^t(\xi), & x \leq \xi, \\ -v_\lambda(x)[W^t(u_\lambda, v_\lambda)]^{-1} u_\lambda^t(\xi), & x > \xi, \end{array} \right.$$

is the Green matrix for the problem (12).

Let us consider the $2n \times 2n$ matrix $\widetilde{W}$ which can be found in the middle of equalities (10). The space $\pi_1(x)$ ($\pi_2(x)$) spanned by the first (respectively, last) n columns of $\widetilde{W}$ is a Lagrangian plane, and therefore $\det W(u_\lambda, v_\lambda) = \det \widetilde{W}$ (it also follows from (13)). Thus, $\det W(u_\lambda, v_\lambda) = 0$ if and only if $\pi_1(x)$ and $\pi_2(x)$ have a nontrivial intersection, in particular, for $x = x_2$. The latter is equivalent to the existence of a nontrivial solution of (12). The first statement of the Lemma is proved.

In order to prove the second statement of the Lemma one needs only to show that the matrix $G_\lambda(x,\xi)$ is continuous at $x = \xi$ and its derivative has a jump equal to $-I$. If π is a Lagrangian plane then planes π and $J\pi$ are orthogonal. From here and (11) it follows that $W(u_\lambda, u_\lambda)$ and $W(v_\lambda, v_\lambda)$ are zero matrices, and

$$\left(\widetilde{W}\right)^t J\widetilde{W} = \left[\begin{array}{cc} 0 & -W^t(u_\lambda, v_\lambda) \\ W(u_\lambda, v_\lambda) & 0 \end{array} \right]. \tag{13}$$

This implies that

$$\widetilde{W} \left[\begin{array}{cc} 0 & -[W(u_\lambda, v_\lambda)]^{-1} \\ -[W^t(u_\lambda, v_\lambda)]^{-1} & 0 \end{array} \right] \left(\widetilde{W}\right)^t = -J.$$

The first column of this matrix relation gives the necessary properties of G_λ at $x = \xi$. The Lemma is proven.

3. Spectral measure. We still assume that the Birman condition (3) holds. As it was mentioned earlier, this implies the boundedness of the spectrum of H from below:

Lemma 2 *If the Birman condition (3) holds then there exists* $\Lambda_0 > -\infty$ *such that* $\Sigma(H) \subset [\Lambda_0, \infty)$.

Proof. Obviously, it is sufficient to prove that

$$(H\varphi, \varphi) \geq \Lambda_0 > -\infty$$

for all φ such that

$$\varphi \in C_0^\infty(R_+), \quad \dot{\varphi}(0) = 0, \quad \|\varphi\|_{L_2} = 1.$$

From the standard Neumann-Dirichlet estimation it follows that Λ_0 can only increase if we allow φ to have jumps at integer points $x = n \geq 0$, but impose the Neumann b.c. at those points. To be more exact, it is enough to show that

there exists $\Lambda_0 > -\infty$ such that, for any n and any smooth function φ on the interval $\Delta_n = (n, n+1)$,

$$(H_n\varphi, \varphi) := \int_n^{n+1} [(\dot{\varphi}, \dot{\varphi}) + (v\varphi, \varphi)]dx \geq \Lambda_0 \tag{14}$$

if

$$\dot{\varphi}(n) = \dot{\varphi}(n+1) = 0, \quad \|\varphi\|_{L_2(\Delta_n)} = 1.$$

Since $\|\varphi\|_{L_2(\Delta_n)} = 1$, there exists a point $x_0 \in \Delta_n$ such that $|\varphi(x_0)| = 1$. Then

$$|\varphi(x)|^2 - |\varphi(x_0)|^2 = 2\int_{x_0}^{x} (\varphi, \dot{\varphi})dz,$$

and therefore, for any $\varepsilon > 0$,

$$|\varphi(x)|^2 \leq 1 + 2\int_n^{n+1} |(\varphi, \dot{\varphi})|dz \leq 1 + \varepsilon \|\varphi\|^2_{L_2(\Delta_n)} + \varepsilon^{-1} \|\dot{\varphi}\|^2_{L_2(\Delta_n)}.$$

Now,

$$\begin{aligned} \left|\int_n^{n+1} (v\varphi, \varphi)dx\right| &\leq \int_n^{n+1} \|v(z)\| \, |\varphi(z)|^2 \, dz \\ &\leq (1 + \varepsilon + \varepsilon^{-1} \|\dot{\varphi}\|^2_{L_2(\Delta_n)}) \int_n^{n+1} \|v(z)\| \, dz. \end{aligned}$$

Finally,

$$(H_n\varphi, \varphi) \geq \|\dot{\varphi}\|^2_{L_2(\Delta_n)} - (1 + \varepsilon + \varepsilon^{-1} \|\dot{\varphi}\|^2_{L_2(\Delta_n)}) \int_n^{n+1} \|v(z)\| \, dz.$$

This inequality with

$$\varepsilon = \int_n^{n+1} \|v(z)\| \, dz$$

implies (14) with

$$\Lambda_0 = -(1 + \int_n^{n+1} \|v(z)\| \, dz).$$

The proof is completed.

After the boundedness of the spectrum of H is established, the general theory provides the essential self adjointness of H, analyticity of the resolvent $R_z = (H - z)^{-1}$, $z \notin [\Lambda_0, \infty)$, the existence, for any element $\varphi \in L^2(R_+)$, of the spectral measures μ_φ such that

$$(R_z\varphi, \varphi) = \int_{\Lambda_0}^{\infty} \frac{\mu_\varphi(d\lambda)}{\lambda - z},$$

etc. This paper concerns a very special spectral measure $\mu(d\lambda)$ associated to the generalized Fourier transform defined by operator H. This measure contains

information about all spectral measures $\mu_\varphi(d\lambda)$ and describes the spectral type of H.

The generalized Fourier transform F for vector functions $\varphi \in L_2(R_+)$ is defined by the formula

$$\widehat{\varphi}(\lambda) = F(\varphi(x)) = \int_0^\infty u_\lambda^t(x)\varphi(x)dx. \tag{15}$$

where u_λ is the matrix solution of the problem

$$(H-\lambda)u_\lambda = 0, \quad x > 0; \quad u_\lambda(0) = I, \quad \dot{u}_\lambda(0) = 0.$$

There exists a unique (due to self adjointness of H) spectral measure $\mu(d\lambda)$ such that the inverse Fourier transform is given by the formula

$$\varphi(x) = \int_{\Lambda_0}^\infty u_\lambda(x)\widehat{\varphi}(\lambda)\mu(d\lambda)$$

and Parseval's identity holds

$$(\varphi_1(x),\ \varphi_2(x)) = \int_{\Lambda_0}^\infty (\widehat{\varphi_1}(\lambda), \widehat{\varphi_2}(\lambda))\mu(d\lambda).$$

One has to treat integrals here as L_2-limits of integrals over the finite intervals (Λ_0, L) as $L \to \infty$. The mapping $\varphi \to \widehat{\varphi}$ is the isomorphism between $L_2(R_+, dx)$ and $L_2(R,\ \mu(d\lambda))$. The spectral measures $\mu_\varphi(d\lambda)$ of elements $\varphi \in L_2(R_+)$ can be expressed through $\mu(d\lambda)$ by the formula

$$\mu_\varphi(d\lambda) = |\widehat{\varphi}(\lambda)|^2 \mu(d\lambda). \tag{16}$$

One of the methods (attributed to B. Levitan) to prove the statements above on the generalized Fourier transform and the spectral measure is based on the weak convergence $\mu_k \to \mu$ of the discrete measures μ_k associated to the restriction of the Hamiltonian H to the set of functions on the interval $(0, k) \subset R_+$ with a Lagrangian b.c. at the point $x = k$. This method is very convenient if one wants to prove that the spectral measure μ is discrete. In order to describe the a.c. component of μ it is better to approximate μ by a.c. measures μ_k. One of such approximations (going back to M. Krein and his school) is based on the averaging of the measures μ_k with respect to the Lagrangian planes related to the b.c. at $x = k$. We use a different approximation of the spectral measure μ.

Let v_s be a sequence of symmetric matrix potentials such that v_s is supported on $[0, s]$ and

$$\int_0^s ||v_s(x) - v(x)||dx \to 0 \quad \text{as} \quad s \to \infty.$$

We denote by H_s the Hamiltonian on $L_2(R_+)$ with the potential v_s and the Neumann b.c. at $x = 0$. Let $\mu_s(d\lambda)$ be the spectral measure of the operator H_s. This measure is a.c. on $[0, \infty)$ (with multiplicity n) and has at most a finite number of atoms on the negative semiaxis at eigenvalues $\lambda_{j,s} < 0$, $j \leq m(s)$, of the operator H_s.

Lemma 3 *Spectral measures $\mu_s(d\lambda)$ converge weakly to $\mu(d\lambda)$ as $s \to \infty$.*

The proof of the Lemma is based on the convergence of the resolvent $R_z^{(s)} = (H_s - z)^{-1}$ and of the corresponding Weil's function when $\mathrm{Im}z > 0$, $s \to \infty$. Similar arguments were used in [7] in a different setting. We shall provide the technical details of the proof of this lemma elsewhere.

4. Operators with smooth compactly supported potentials. We assume in this section that $v \in C_0^\infty(R_+)$. Let $u_\lambda = u_\lambda(x)$ be the matrix solution of the problem

$$(H - \lambda)u = 0, \quad x > 0; \quad u(0) = I, \quad \dot{u}(0) = 0,$$

and let v_λ^+, v_λ^- be matrix solutions of the some equation such that $v_\lambda^\pm(x) = e^{\pm i\sqrt{\lambda}x}I$ as $x >> 1$. If $\mathrm{Im}\lambda > 0$ then elements of the matrix v_λ^+ (v_λ^-) decay exponentially as $x \to \infty$ ($x \to -\infty$) and they grow exponentially as $x \to -\infty$ ($x \to \infty$). The columns of the matrices v_λ^+ and v_λ^- taken together form a basis in the solution space of the equation $(H_s - \lambda)u = 0$. Thus,

$$u_\lambda(x) = v_\lambda^+(x)A(\lambda) + v_\lambda^-(x)B(\lambda), \tag{17}$$

where matrices $A(\lambda)$, $B(\lambda)$ are analytic in λ (with a branch point at $\lambda = 0$), and

$$A = \overline{B} \quad \text{for } \lambda > 0, \tag{18}$$

since u_λ is real when $\lambda > 0$. Operator H has at most a finite number of eigenvalues λ_j, $j \leq m$, and $\lambda_j < 0$.

Lemma 4 *The spectral measure $\mu(d\lambda)$ of the operator H is equal to*

$$\mu(d\lambda) = \sum_{j=1}^{m} \delta(\lambda - \lambda_j)d\lambda + \frac{\theta(\lambda)}{4\pi\sqrt{\lambda}}N(\lambda)d\lambda, \quad N(\lambda) = [B(\lambda)]^{-1}[\overline{B^t}(\lambda)]^{-1} \ ,$$

where $\theta(\lambda) = 1$ for $\lambda \geq 0$, $\theta(\lambda) = 0$ for $\lambda < 0$.

Proof. We shall use below the following obvious properties of the Wronskians. If $u = u(x)$, $v = v(x)$ and Q are arbitrary $n \times n$ matrices, and Q does not depend on x, then

$$W(u,u) = 0, \quad W(u, vQ) = Q^t W(u,v), \quad W(uQ, v) = W(u,v)Q. \tag{19}$$

Note that

$$W(v^+, v^+) = 0, \quad W(v^-, v^+) = -W(v^+, v^-) = 2i\sqrt{\lambda}I. \tag{20}$$

In order to justify these relations one can evaluate the Wronskians for $x >> 1$ and use their independence of x. Relations (17), (19) and (20) immediately imply that

$$W(u_\lambda, v^+) = -2i\sqrt{\lambda}A, \quad W(u_\lambda, v_\lambda^-) = 2i\sqrt{\lambda}B, \tag{21}$$

and

$$0 = W(u_\lambda, u_\lambda) = W(v^+A, v^-B) + W(v^-B, v^+A) = 2i\sqrt{\lambda}[A^tB - B^tA].$$

Thus,

$$A^tB = B^tA. \tag{22}$$

If $G = G_\lambda(x,\xi)$ is Green's matrix of the problem

$$(H-\lambda)G = \delta(x-\xi)I, \quad x > 0, \quad \mathrm{Im}\lambda \neq 0; \quad \dot{G}(0,\xi) = G(\infty,\xi) = 0,$$

then from (21) and Lemma 1 it follows that, for $x < \xi$,

$$\begin{aligned} G_\lambda(x,\xi) &= \frac{-1}{2i\sqrt{\lambda}}u_\lambda(x)B^{-1}[v^+(\xi)]^t, \quad \mathrm{Im}\lambda > 0, \\ G_\lambda(x,\xi) &= \frac{1}{2i\sqrt{\lambda}}u_\lambda(x)A^{-1}[v^-(\xi)]^t, \quad \mathrm{Im}\lambda < 0. \end{aligned}$$

One could also write the corresponding formulas when $x > \xi$. Thus, if $\lambda > 0$ and $x < \xi$, then (22) and (17) imply

$$\begin{gathered} G_{\lambda+i0}(x,\xi) - G_{\lambda-i0}(x,\xi) \\ = \frac{-1}{2i\sqrt{\lambda}}u_\lambda(x)B^{-1}(A^{-1})^tA^t[v^+(\xi)]^t - \frac{1}{2i\sqrt{\lambda}}u_\lambda(x)A^{-1}(B^{-1})^tB^t[v^-(\xi)]^t \\ = \frac{-1}{2i\sqrt{\lambda}}u_\lambda(x)B^{-1}(A^{-1})^tu_\lambda(\xi). \end{gathered}$$

The same formula is valid when $x > \xi$. One can derive it similarly, but the easier way to get it is to note that the operator with the kernel $i[G_{\lambda+i0} - G_{\lambda-i0}]$ is symmetric. This and (18) imply

$$G_{\lambda+i0}(x,\xi) - G_{\lambda-i0}(x,\xi) = \frac{1}{2i\sqrt{\lambda}}u_\lambda(x)B^{-1}(\overline{B}^{-1})^tu_\lambda(\xi), \qquad \lambda > 0.$$

To complete the proof of Lemma 4 it remains only to apply the Stone formula and use (16).

Let us denote by $w = w_\lambda(x)$ the matrix *Jost solution* for operator H. This solution is defined by the relations

$$(H-\lambda)w = 0, \quad x > 0; \quad w(0) = I, \quad \dot{w}(0) = -i\sqrt{\lambda}I \ .$$

Similarly to (17) we have

$$w_\lambda(x) = v_\lambda^+(x)P(\lambda) + v_\lambda^-(x)S(\lambda), \tag{23}$$

where matrices $P(\lambda)$ and $S(\lambda)$ are analytic in λ with a branch point at $\lambda = 0$. If the Jost solution is extended by $e^{-i\sqrt{\lambda}x}I$ for $x < 0$ and H' is the Hamiltonian H on the whole axis with the potential v extended by zero for $x < 0$,

then $[S(\lambda)]^{-1}w_\lambda(x)$ is the *scattering solution* for the operator H'. It describes the propagation of the incident plane wave $e^{-i\sqrt{\lambda}x}I$ coming from $x=\infty$ with $[S(\lambda)]^{-1}$ being the transmission matrix and $P(\lambda)[S(\lambda)]^{-1}$ being the reflection matrix. We shall call $S(\lambda)$ the *Jost transmission matrix.* Note that (23) for large enough x can be written in the form

$$w_\lambda(x) = e^{i\sqrt{\lambda}x}P(\lambda) + e^{-i\sqrt{\lambda}x}S(\lambda), \quad x >> 1. \tag{24}$$

The following theorem allows us to estimate the density of the spectral measure $\mu(d\lambda)$ through the Jost transmission matrix which has better asymptotic behavior for complex $\lambda \to \infty$ than $B(\lambda)$.

Theorem 5 *The following estimates hold for the matrix $N(\lambda)$*

$$\frac{1}{4} \le ||[N(\lambda)]^{-1}|| \le ||S(\lambda)||^2 \le |\det S(\lambda)|^2.$$

Proof. Green's formula for the columns of the matrix $w = w_\lambda(x)$ can be written in the form

$$0 = \int_0^a [(Hw)^t \ddot{w} - \ddot{w}^t Hw]dx = 2i\mathrm{Im}[(\dot{w}^t\overline{w})(a) - (\dot{w}^t\overline{w})(0)].$$

Thus,

$$\mathrm{Im}(\dot{w}^t\overline{w})(x) = \sqrt{\lambda}I, \quad x > 0. \tag{25}$$

We choose s so big that (24) holds for $x > s$. If we substitute (24) into (25), take the average of both sides of (25) over interval $(s, s+l)$ and pass to the limit as $l \to \infty$, then we arrive to the following relation ("conservation of the energy"):

$$\overline{S}^t(\lambda)S(\lambda) - \overline{P}^t(\lambda)P(\lambda) = I, \quad \lambda > 0. \tag{26}$$

In particular, from (26) it follows that $||S\varphi|| \ge ||\varphi||$ for any vector φ, and therefore,

$$|\mu_j(\lambda)| \ge 1, \quad \lambda > 0, \tag{27}$$

where $\mu_j(\lambda)$ are eigenvalues of the matrix $S(\lambda)$. Formulas (26) and (27) imply that

$$1 + ||P|| \le ||S|| \quad \text{and } ||S|| \le |\det S|, \quad \lambda > 0. \tag{28}$$

Obviously, $u_\lambda = \frac{1}{2}(w_\lambda + \overline{w}_\lambda)$, $\lambda > 0$. Thus, from (17) and (23) it follows that

$$B = \frac{1}{2}(S + \overline{P}), \quad \lambda > 0.$$

From here and (28) it follows that

$$||\overline{B}^t B|| = \frac{1}{4}||S + \overline{P}||^2 \le ||S||^2 \le |\det S|,$$

and

$$||\overline{B}^t B|| = \frac{1}{4}||S + \overline{P}||^2 \ge \frac{1}{4}(||S||^2 - ||\overline{P}||)^2 \ge \frac{1}{4}.$$

The proof is completed.

We shall need an asymptotic expansion of $\det S(\lambda)$ at infinity. Let us recall that $S(\lambda)$ is analytic in the complex λ-plane $\mathbf{C}$ with a branch point at the origin. Let $\mathbf{C}_1 = \mathbf{C}\backslash[0,\infty)$.

Let $P = P(v, \dot{v}, ...)$ be a polynomial of v and its derivatives. Note that terms of P depend on the order of their factors. We shall say that P is generalized homogeneous of order m if the substitution $v^{(l)}(x) \to \varepsilon^{2+l}v^{(l)}(x)$ in the arguments of P results in multiplication of P by ε^m.

Theorem 6 *If the matrix potential v belongs to $C_0^\infty(R_+)$ then the following expansion is valid*

$$\ln[\det S(\lambda)] \sim i\sum_{m=0}^{\infty}\frac{I_m}{\lambda^{m+1/2}}, \quad \lambda \in \mathbf{C}_1, \quad |\lambda| \to \infty, \tag{29}$$

where $I_m = I_m(v)$ are functionals of v of the form

$$I_m = \int_0^\infty P_m(v, \dot{v}, ...)dx. \tag{30}$$

Here P_m are generalized homogeneous polynomials of v and its derivatives, and the order of P_m is $2m+2$.

Remarks. 1). Polynomials P_m are not defined uniquely. For example, one can add $\dot{v}$ to any P_m. After polynomials P_m are found, one can change them in such a way (by integrating by parts in (30)) that P_m will depend on derivatives of v of the order at most $m-1$.

2). The first polynomials P_m are:

$$P_0 = v, \quad P_1 = v^2, \quad P_2 = \frac{1}{2}\dot{v}^2 + v^3, \quad P_3 = \frac{1}{2}\ddot{v}^2 - \frac{5}{4}v^2\ddot{v} - \frac{5}{4}\ddot{v}v^2 + \frac{5}{4}v^4.$$

3). Theorem 6 is well known in the scalar case. In this case, I_m are first integrals of the KdV equation.

Proof. Put $w = e^{-i\sqrt{\lambda}x}z$. Then

$$-\ddot{z} + 2i\sqrt{\lambda}\dot{z} + v(x)z = 0, \quad x \in R; \quad z = I \ \text{ for } \ x < 0. \tag{31}$$

We denote $\dot{z}z^{-1}$ by $Q(x,\lambda)$. Then $\dot{z} = Qz$ and $\ddot{z} = \dot{Q}z + Q\dot{z} = (\dot{Q} + Q^2)z$. From here and (31) it follows that

$$-\dot{Q} - Q^2 + 2i\sqrt{\lambda}Q + v(x) = 0, \quad x \in R; \quad Q = 0 \ \text{ for } \ x < 0.$$

Let $v = 0$ for $x > s$. If $|x| \le s+1$, $\lambda \in \mathbf{C}_1$, $|\lambda| \to \infty$, then solution Q of the above matrix equation can be easily found in the form of the power series:

$$Q \sim \sum_1^\infty (2i\sqrt{\lambda})^{-m}Q_m(x), \quad |x| \le s+1, \ \lambda \in \mathbf{C}_1, \ |\lambda| \to \infty, \tag{32}$$

where

$$Q_1 = v, \quad Q_2 = \dot{v}, \quad Q_m(x) = \dot{Q}_{m-1} + \sum_{k=1}^{m-2} Q_k \, Q_{m-k-1}, \quad m > 2.$$

One can show by induction that Q_m are generalized homogeneous polynomials of v and its derivatives, and $\mathrm{ord} Q_m = m+1$.

In order to find z for $|x| \le s+1$ one has to solve the following matrix equation

$$\dot{z} = Qz, \quad x \in R; \quad z = I \ \text{ for } \ x < 0. \tag{33}$$

We shall forget temporary about dependence of Q on λ. The solution of (33) can be written in the form of the matrix multiplicative integral:

$$z(x) = \prod_{t=1}^{x} (I + Q(t)dt), \tag{34}$$

which is a short notation for the following limit

$$z(x) = \lim_{\max|\Delta x_i| \to 0} \prod_{i=1}^{m} (I + Q(x_i)\Delta x_i). \tag{35}$$

Here x_i, $0 \le i \le m$, are points on $[0,x]$, $x_i < x_{i+1}$, $x_0 = 0$, $x_m = x$, and $\Delta x_i = x_{i+1} - x_i$. Thus, one arrives to the multiplicative integral (34) if (33) is solved by the Euler method. From (34), (35) it follows that

$$\det z(x) = \prod_{t=1}^{x} (I + \mathrm{tr} Q(t)dt).$$

The last expression is the solution of the scalar equation

$$\dot{y} = [\mathrm{tr} Q(x)]y, \quad x \in R; \quad y(x) = 1 \ \text{ for } \ x < 0.$$

Thus,

$$\det z(x) = e^{\int_0^\infty \mathrm{tr} Q(t)dt}. \tag{36}$$

Since Q_m are polynomials of v and its derivatives, then $Q_m = 0$ for $x > s$, that is $Q = O(|\lambda|^{-\infty})$ when $s \le x \le s+1$, $\lambda \in \mathbf{C}_1$, $|\lambda| \to \infty$. Then (33) implies that

$$z(x,\lambda) = z_0(\lambda) + O(|\lambda|^{-\infty}), \quad s \le x \le s+1, \ \lambda \in \mathbf{C}_1, \ |\lambda| \to \infty.$$

This and (24) imply that

$$e^{i\sqrt{\lambda}x} P(\lambda) + e^{-i\sqrt{\lambda}x}[S(\lambda) - z_0(\lambda)] = e^{-i\sqrt{\lambda}x} O(|\lambda|^{-\infty}).$$

We substitute here two different values for x from the interval $(s, s+1)$ and solve the system for $P(\lambda)$, $S(\lambda) - z_0(\lambda)$. This leads to

$$S(\lambda) = z_0(\lambda) + O(|\lambda|^{-\infty}), \quad \lambda \in \mathbf{C}_1, \ |\lambda| \to \infty.$$

Hence,

$$\det S(\lambda) = \det z(x,\lambda) + O(|\lambda|^{-\infty}) \quad \text{for any } x \in (s, s+1),$$

and from (36), (32) it follows that

$$\ln[\det S(\lambda)] \sim \sum_{m=1}^{\infty} [(2i\sqrt{\lambda})^{-m} \int_0^s Q_m(x)dx] \sim \sum_{m=1}^{\infty} [(2i\sqrt{\lambda})^{-m} \int_0^{\infty} Q_m(x)dx] \tag{37}$$

as $\lambda \in \mathbf{C}_1$, $|\lambda| \to \infty$. In order to complete the proof of Theorem 6 it remains only to show that the terms with even m in the formula above are zeroes. It will be done in the process of proving the next theorem.

The following trace type theorem (compare [9], formula (34)) allows us to estimate the density of the spectral measure of the operator H through the coefficients I_m in the asymptotic expansion (29).

Theorem 7 *For any $m \geq 0$,*

$$\frac{1}{\pi} \int_0^{\infty} \lambda^{m-1/2} \ln|\det S(\lambda)| d\lambda = I_m + (-1)^m \sum_{j=1}^{q} \frac{2|\lambda_j|^{m+1/2}}{2m+1}. \tag{38}$$

Here $\lambda_j < 0$, $1 \leq j \leq q$, are eigenvalues of H.

Proof. It will be convenient for us to introduce $z = \sqrt{\lambda}$. Then the upper half plane Im$z > 0$ corresponds to $\mathbf{C}' = \mathbf{C} \backslash [0, \infty)$, and points $z_j = ik_j$, $k_j = \sqrt{|\lambda_j|}$, on the positive part of imaginary axis correspond to the eigenvalues of H. Let $B(z)$ be the Blashke product:

$$B(z) = \prod_{j=1}^{q} \frac{z - ik_j}{z + ik_j}.$$

Then the function

$$\ln \frac{\det S(z^2)}{B(z)}$$

is analytic in the upper half plane. Since $|B(z)| = 1$ for real z, then

$$\mathrm{Re}[\ln \frac{\det S(z^2)}{B(z)}] = \ln|\det S(z^2)|, \quad z \in R,$$

and by the Herglotz formula

$$\frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{\ln|\det S(k^2)|}{z - k} dk = \ln[\det S(z^2)] - \ln B(z), \quad \mathrm{Im} z \geq 0. \tag{39}$$

We are going to write the asymptotic expansion for all terms in (39) as Im$z \geq 0$, $|z| \to \infty$. The expansion of the first term in the right-hand side is given by (37). Obviously,

$$\ln \frac{z - ik}{z + ik} \sim i \sum_{m=1}^{\infty} \frac{2(-1)^{m+1} k^{2m+1}}{(2m+1) z^{2m+1}}, \quad \mathrm{Im} z \geq 0, \quad |z| \to \infty.$$

This leads to the asymptotic expansion for the second term in the right-hand side of (39). One has to be careful when $S(k^2)$ is considered for real $k = k_0$. The values of $S(k_0^2)$ are understood as $S((k_0 + i0)^2)$. One can easily show using (24), that $S(\lambda + i0) = \overline{S}(\lambda - i0)$ for $\lambda > 0$, i.e. $S(k_0^2)$ is an even function. Hence, if Im$z \geq 0$, $|z| \to \infty$, then

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{\ln|\det S(k^2)|}{z-k} dk &\sim \sum_{m=1}^{\infty} \frac{1}{z^{2m+1}} \int_{-\infty}^{\infty} k^{2m} \ln|\det S(k^2)| dk \\ &\sim \sum_{m=1}^{\infty} \frac{1}{z^{2m+1}} \int_{0}^{\infty} \lambda^{m-1/2} \ln|\det S(\lambda)| d\lambda.\end{aligned}$$

We equate the coefficients in the asymptotic expansions of the left and right-hand sides of (39) as Im$z \geq 0$, $|z| \to \infty$. The corresponding equalities for odd powers of $1/z$ imply (38), and the equalities for even powers of $1/z$ show that the terms in (37) with even m are zeroes. The proofs of Theorems 6 and 7 are completed.

The following statement is an obvious consequence of Lemma 4 and Theorems 5 and 7.

Theorem 8 *Let $\mu_{ac}(d\lambda) = h(\lambda)d\lambda$ be the a.c component of the spectral measure of the operator H with a matrix potential $v \in C_0^\infty(R)$, and let $\delta = [\lambda_1, \lambda_2]$, $\lambda_2 > \lambda_1 > 0$, be an interval of the positive semiaxis. Then for any $m \geq 0$*

$$||[h(\lambda)]^{-1}|| \geq \pi\sqrt{\lambda_1}, \quad \lambda \in \delta,$$

and

$$\int_\delta \ln||[h(\lambda)]^{-1}|| d\lambda \leq \pi\lambda_2^{m-1/2}\{|I_m| + \sum_{j=1}^{q} \frac{2|\lambda_j|^{m+1/2}}{2m+1}\}.$$

The last statement, which will be proven in this section, is devoted to the generalization of the Lieb-Thirring estimates to the matrix case. It will allow us to simplify the estimate in Theorem 8.

Lemma 9 *Let $\{\lambda_j\}$ be the set of negative eigenvalues for the Schrödinger operator with a matrix potential v, and let*

$$v_\gamma := \int_0^\infty ||v(x)||^{\gamma+1/2} dx < \infty.$$

Then

$$\sum_j |\lambda_j|^\gamma \leq n v_\gamma,$$

where $n \times n$ is the size of the matrix v.

Proof. Let $\mu(x) = ||v(x)||$ and let $H_- = -\frac{d^2}{dx^2} - \mu(x)I$ be the matrix Schrödinger operator on $L_2(R_+)$ with the same b.c. at $x = 0$ as for operator

H. Then $H_- \leq H$ and $\lambda_i^- \leq \lambda_i$, where λ_i^- are eigenvalues of H_- and the eigenvalues $\{\lambda_i^-\}$ and $\{\lambda_i\}$ are numerated in increasing order. Thus,

$$\sum_j |\lambda_j|^\gamma \leq \sum_j |\lambda_j^-|^\gamma.$$

It remains only to note that $\{\lambda_i^-\}$ are eigenvalues of the scalar Schrödinger operator repeated n times.

Let us recall that the functional J_m (see (2)) has the same form as the functional I_m (see (30)) with the integrand in (2) being a generalized homogeneous of order $2m+2$ function of v and its derivatives. There exists an independent of v constant c_m such that

$$|I_m| \leq c_m J_m.$$

This estimate was proven in the scalar case in [9] using Kolmogorov type estimates, and the same proof remains valid in the matrix case. The last estimate together with Lemma 9 allow us to rewrite Theorem 8 in the following form.

Theorem 10 *Let $\mu_{ac}(d\lambda) = h(\lambda)d\lambda$ be the a.c component of the spectral measure of the operator H with a matrix potential $v \in C_0^\infty(R)$, and let $\delta = [\lambda_1, \lambda_2]$, $\lambda_2 > \lambda_1 > 0$, be an interval of the positive semiaxis. Then there exist independent of v constants $c(\delta) > 0$ and $C(\delta)$ such that for any $m \geq 0$*

$$||[h(\lambda)]^{-1}|| \geq c(\delta), \quad \lambda \in \delta, \quad \textit{and} \quad \int_\delta \ln ||[h(\lambda)]^{-1}|| d\lambda \leq C(\delta) J_m(v).$$

5. Generalized L_2-conjecture. We shall need the following

Lemma 11 *Let $v = v(x)$ be a matrix potential such that*

$$J_p(v) < \infty$$

for some $p \geq 0$, where J_p is the functional defined in (2). Then there is a sequence of matrix potentials $v_s \in C_0^\infty(R_+)$ such that $v_s(x) = 0$ for $x > s$ and

$$J_p(v - v_s) \to 0, \qquad \int_0^s ||v(x) - v_s(x)|| dx \to 0 \quad \textit{as} \quad s \to \infty. \tag{40}$$

Proof. We fix a function $\varphi_s(x) \in C^\infty(R)$, such that $\varphi_s(x) = 1$ for $x < s-1$, $\varphi_s(x) = 0$ for $x > s - 1/2$. The boundedness of J_p implies that

$$\int_{s-1}^\infty (||v^{(p-1)}(x)||^2 + ||v(x)||^{p+1}) dx \to 0 \quad \text{as} \quad s \to \infty.$$

From here it follows that relations (40) hold for the matrix $v - \varphi_s v$. Let $\alpha = \alpha(x)$ be a $C_0^\infty(R)$ -function with the support strictly inside of the interval $[0, 1]$. Let

$$\alpha_\varepsilon(x) = \varepsilon^{-1}\alpha(\varepsilon x) / \int_0^1 \alpha(x) dx.$$

The convolution $v_{s,\varepsilon} = \varphi_s v * \alpha_\varepsilon$ is a $C_0^\infty(R)$-function supported on the interval $(0,s)$ if $\varepsilon < 1/2$. Since

$$J_p(\varphi_s v - v_{s,\varepsilon}) \to 0, \qquad \int_0^s ||\varphi_s v(x) - v_{s,\varepsilon}(x)||dx \to 0 \quad \text{as} \ \ \varepsilon \to 0, \tag{41}$$

then one can choose $\varepsilon = \varepsilon(s) < 1/2$ to be so small that the left-hand sides of (41) are less than $1/s$. Then $v_s = v_{s,\varepsilon(s)}$ satisfies the requirements of Lemma 11. The proof is completed.

The following criteria of the absolute continuity of the limit measure was proven in [9].

Theorem 12 *Let $\mu_s(d\lambda) = h_s(\lambda)d\lambda$ be a sequence of absolutely continuous positive (scalar) measures on an interval $\delta = [\lambda_1, \lambda_2]$. Let $\mu_s(d\lambda)$ converge weakly to a measure $\mu(d\lambda)$ as $s \to \infty$, and for any s*

$$\int\limits_{\delta \cap \{h(\lambda)<1\}} \ln \frac{1}{h_s(\lambda)} d\lambda < C < \infty, \tag{42}$$

where C does not depend on s. Then the essential support of the a.c. component $\mu_{ac}(d\lambda)$ of the limit measure coincides with δ.

Remarks 1. If $h_s(\lambda) < c < \infty$ then one can integrate over δ in (42), since the integral over $\delta \cap \{h(\lambda) > 1\}$ is uniformly bounded in this case.

2. It was also shown in [9] that the statement of the theorem remains valid if the logarithmic function above is replaced by any function monotonically increasing on $[0,\infty]$ (we shall not need this fact).

We shall say that an a.c. $n \times n$ matrix measure $\mu(d\lambda) = \nu(\lambda)d\lambda$ has the multiplicity n and is essentially supported on some interval $\delta = [\lambda_1, \lambda_2]$ if the latter is true for all scalar measures $\nu_j(\lambda)d\lambda$ where $\nu_j(\lambda)$ are eigenvalues of the matrix $\nu(\lambda)$. It is easy to see that the following matrix analogue of Theorem 12 can be immediately reduced to a scalar case.

Theorem 13 *Let $\mu_s(d\lambda) = h_s(\lambda)d\lambda$ be a sequence of absolutely continuous positive matrix measures on an interval $\delta = [\lambda_1, \lambda_2]$. Let $\mu_s(d\lambda)$ converge weakly to a measure $\mu(d\lambda)$ as $s \to \infty$, and for any s*

$$||[h_s(\lambda)]^{-1}|| \geq c(\delta) > 0, \quad \lambda \in \delta; \qquad \int_\delta \ln ||[h_s(\lambda)]^{-1}||d\lambda < C(\delta) < \infty, \tag{43}$$

where c, C do not depend on s. Then the a.c. component $\mu_{ac}(d\lambda)$ of the limit measure has multiplicity n and is essentially supported on δ.

The following theorem presents the main result.

Theorem 14 *Let H be a Schrödinger operator with a matrix potential $v = v(x)$ such that*

$$J_p(v) < \infty$$

for some $p \geq 0$. Then the spectral measure $\mu(d\lambda)$ of the operator H has the a.c. component of the multiplicity n which is essentially supported on $[0, \infty)$.

Proof. Let $\{v_s\}$ be a sequence of $C_0^\infty(R_+)$ -potentials constructed in Lemma 11 and let $\mu_s(d\lambda)$ be the spectral measure of the operator H_s with the potential v_s. Lemma 3 and the second of relations (40) imply the weak convergence of $\mu_s(d\lambda)$ to $\mu(d\lambda)$ as $s \to \infty$. Hence, the convergence holds for restrictions of these measures on the semiaxis $\lambda > 0$, where measures $\mu_s(d\lambda)$ are a.c. (see Lemma 4). Due to the first of relations (40), $J_p(v_s) \leq C = 2J_p(v)$ if s is big enough. Hence, Theorem 14 is an immediate consequence of Theorems 10 and 13.

References

[1] F. V. Atkinson, Discrete and Continuous Boundary Problems, Academic Pres, N.-Y.,1064.

[2] R. Carmona, J. Lacroix, Spectral Theory of Random Schrodinger Operator, Birhauser Verlag, Basel,Boston, Berlin, 1990.

[3] E. Coddington, N. Levinson, Theory of Ordinary Differential Equations, McGrow-Hill, 1955.

[4] P. Deift, R. Killip, On the absolutely continuous spectrum of one-dimensional Schrödinger operators with square summable potentials, Comm. Math. Phys., 203, 341- 347 (1999).

[5] F. Delyon, B. Simon, B. Souillard, From pure point to continuous spectrum in disordered systems, Ann. Inst. H. Poincaré, 42 (1985), 283-309

[6] S. Kotani, N. Ushiroya, One-dimensional Schrödinger operators with random decaying potentials, Comm. Math. Phys., 115 (1988), 247-266.

[7] S. Kotani, B. Simon, Stochastic Schrödinger operators and Jacobi matrices on the strip, Comm. Math. Phys., 119, (1988), 403-429.

[8] B. M. Levitan, I. S. Sargsyan, Introduction to Spectral Theory, Translation Math. Monographs,39, AMS, Providence, RI, 1976.

[9] S. Molchanov, M. Novitskii and B. Vainberg, First KdV integrals and absolutely continuous spectrum for 1-D Schrödinger operator, Comm. of Math. Phys., 216, 195-213 (2001).

[10] M. Reed, B. Simon, Methods of Modern Mathematical Physics, III: Scattering Theory, Academic Press, N.-Y., 1979

[11] B. Simon, T. Spencer, Trace class perturbation and the absence of absolutely continuous spectrum, Comm. Math. Phys., 125, (1989), 113-125.